GENERAL BOUNDARY-CONDITION RECALCULATION ALGORITHM FOR TEMPERATURE FIELDS OF DIFFERENT GEOMETRY

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An integral relation, applicable to temperature fields of different geometry, is presented to link the temperature of a surface subjected to heating with the heat flux.

As is well known, in many instances the algorithm for calculating heat fluxes from measurements of the nonsteady temperatures of a surface subjected to thermal action is constructed on the basis of a relation of the type [1]

$$\bar{q}(\mathrm{Fo}) = \int_0^{\mathrm{Fo}} T'(\tilde{\mathrm{Fo}})\, P(\mathrm{Fo} - \tilde{\mathrm{Fo}})\, d\tilde{\mathrm{Fo}}, \tag{1}$$

where the kernel P is determined by the geometry of the structure and in the general case is expressed in terms of series of transcendental functions or combinations of same.

In a numerical realization of (1), such as by means of the algorithm [1]

$$\bar{q}(k\Delta\,\mathrm{Fo}) = A \sum_{j=1}^{k} P_{k-j+1}(T_{j+1} - T_j), \quad k = 1,\ 2,\ \ldots,\ N, \tag{2}$$

it is necessary either to use sets of previously calculated values of P_s for different ΔFo and different temperature-field geometries or to provide for computation of these coefficients each time directly in the program. These requirements, while not fundamentally limiting in computer calculations, nevertheless create certain inconveniences connected with having to calculate P_s for the entire realization in accordance with different algorithms — depending on the type of thermal model — and with the use of different dependences for large and small Fo. On the other hand, it turns out that it is possible to construct boundary-condition recalculation algorithms distinguished by great generality compared to the above and, thus, more convenient to realize in certain instances.

We will examine the relationship in the space of Laplace transforms between q and T for the thermal models shown in Fig. 1. On the basis of [2] for constant thermophysical properties, as well as when they change in the spatial variable according to the laws

$$\lambda(r)\, r^k = \tilde{\lambda}_0 r^l, \quad c(r)\, r^k = \tilde{c}_0 r^n; \quad \lambda(r)\, r^k = \tilde{\lambda}_0 (1 + \alpha r)^l, \quad c(r)\, r^k = \tilde{c}_0 (1 + \alpha r)^n;$$
$$\lambda(r)\, r^k = \tilde{\lambda}_0 \exp(-lr), \quad c(r)\, r^k = \tilde{c}_0 \exp(-nr), \tag{3}$$

we have the following relations:

$$T(r,\ s) = \left[\frac{\tilde{\lambda}(r)\,\tilde{c}(r)}{\tilde{\lambda}(r_0)\,\tilde{c}(r_0)}\right]^{\frac{1}{2(m-2)}} \frac{I_{-\nu}\left[\dfrac{2E\sqrt{s}}{m}\left(\dfrac{\tilde{\lambda}(r)\,\tilde{c}(r)}{E^2}\right)^{\frac{m}{2(m-2)}}\right]}{I_{-\nu}\left[\dfrac{2E\sqrt{s}}{m}\left(\dfrac{\tilde{\lambda}(r_0)\tilde{c}(r_0)}{E^2}\right)^{\frac{m}{2(m-2)}}\right]}\, T(r_0,\ s), \qquad 0 \leqslant r \leqslant r_0 \tag{4}$$

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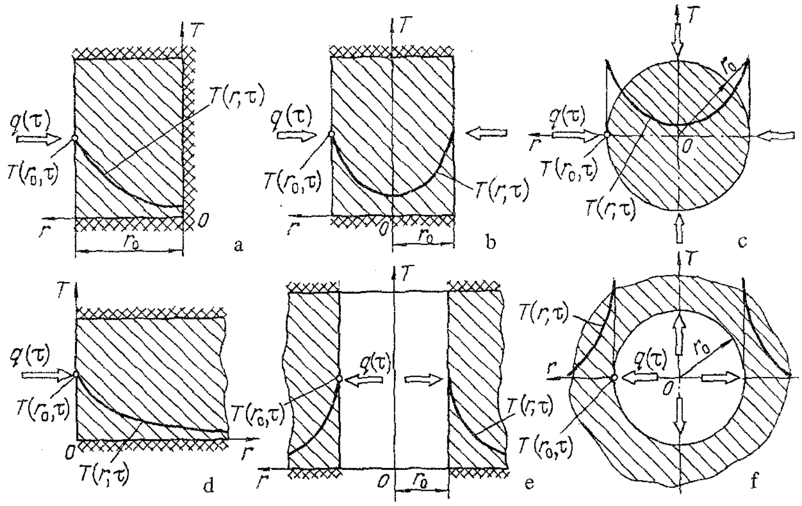

Fig. 1. Diagram of the thermal models: a, d) plane field, k = 0; b, e) cylindrical, k = 1; c, f) spherical, k = 2.

(models a, b, c);

$$T(r,\ s)=\left[\frac{\tilde{\lambda}(r)\,\tilde{c}(r)}{\tilde{\lambda}(r_0)\,\tilde{c}(r_0)}\right]^{\frac{1}{2(m-2)}}\frac{K_{-\nu}\left[\frac{2E\sqrt{s}}{m}\left(\frac{\tilde{\lambda}(r)\,\tilde{c}(r)}{E^2}\right)^{\frac{m}{2(m-2)}}\right]}{K_{-\nu}\left[\frac{2E\sqrt{s}}{m}\left(\frac{\tilde{\lambda}(r_0)\,\tilde{c}(r_0)}{E^2}\right)^{\frac{m}{2(m-2)}}\right]} \tag{5}$$

(models d, e, f);

$$T(r,\ s)=\int_0^\infty T(r,\ \tau)\exp(-sr)\,d\tau,$$

I_ν and K_ν are Bessel and Macdonald functions. Here, $\lambda(r)\,r^k\equiv\tilde{\lambda}(r)$, $c(r)\,r^k\equiv\tilde{c}(r)$, while E, ν, and m for the above laws of change in λ and c are, respectively, equal to

$$\nu=\frac{1-l}{n-l+2},\quad m=\frac{1}{\nu},\quad E=\sqrt{\tilde{c}_0}\,\tilde{\lambda}_0^{\frac{m-1}{2}}(1-l)^{\frac{m-2}{2}};$$

$$\nu=\frac{1-l}{n-l+2},\quad m=\frac{1}{\nu},\quad E=\sqrt{\tilde{c}_0}\,\tilde{\lambda}_0^{\frac{m-1}{2}}\alpha^{\frac{m-2}{2}}(1-l)^{\frac{m-2}{2}};$$

$$\nu=\frac{l}{l-n},\quad m=\frac{1}{\nu},\quad E=\sqrt{\tilde{c}_0}\,\tilde{\lambda}_0^{\frac{m-1}{2}}l^{\frac{m-2}{2}};\quad l<1;\ n\geqslant-1+k.$$

It is apparent from (4) and (5) that there exists the following relationship between $T(r_0,\ s)$ and $q(r_0,\ s)$:

$$\frac{q(r_0,\ s)}{T(r_0,\ s)}=\frac{\lambda(r_0)}{\sqrt{a(r_0)}}\,\frac{I_{-\nu+1}[z(r_0)]}{I_{-\nu}[z(r_0)]}, \tag{6}$$

$$\frac{q(r_0,\ s)}{T(r_0,\ s)}=\frac{\lambda(r_0)}{\sqrt{a(r_0)}}\,\frac{K_{-\nu+1}[z(r_0)]}{K_{\nu}[z(r_0)]}, \tag{7}$$

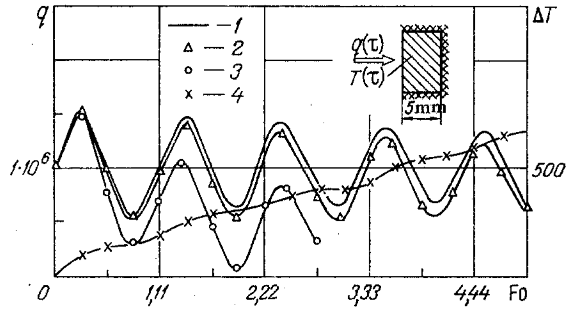

Fig. 2. Results of solution of the model problem: 1) actual values of heat flux q; 2, 3) calculated values of q with Fo* = 0.25 and Fo* = 0.18; 4) temperature of the surface; q, W/m^2; ΔT, °K.

while

$$z(r_0) = 2Em^{-1}\sqrt{s}\,[\tilde{\lambda}(r_0)\tilde{c}(r_0)E^{-2}]^{\frac{m}{2(m-2)}},\quad \lambda(r_0)=\tilde{\lambda}(r_0)\,r_0^{-k},\ c(r_0)=\tilde{c}(r_0)\,r_0^{-k},\ a(r_0)=\lambda(r_0)/c(r_0).$$

After differentiation of (6) and (7) with respect to the Laplace transform parameter, we obtain

$$[q'(s)\,T(s)-q(s)\,T'(s)]-\frac{\nu}{s}\,q(s)\,T(s)+f_1\frac{\sqrt{a(r_0)}\,z_1}{\lambda(r_0)}\,\frac{q^2(s)}{s}=f_1\frac{\lambda(r_0)\,z_1}{\sqrt{a(r_0)}}\,T^2(s). \tag{8}$$

where

$$q(s)\equiv q(r_0,\ s);\quad T(s)\equiv T(r_0,\ s);\quad z_1=\frac{E}{m}\,[\tilde{\lambda}(r_0)\,\tilde{c}(r_0)\,E^{-2}]^{\frac{m}{2(m-2)}},$$

while the parameter f_1 is equal to 1 for models *a*, b, and c and −1 for models d, e, and f.

From (8), after transferring to the originals in accordance with the transformation theorems in [2] and after performing several other transformations, we obtain

$$\int\limits_0^{\mathrm{Fo}}\bar{q}(\mathrm{Fo}-\tilde{\mathrm{Fo}})\,\{f_1[(2\tilde{\mathrm{Fo}}-\mathrm{Fo})\,T(\tilde{\mathrm{Fo}})-\nu\int\limits_0^{\tilde{\mathrm{Fo}}}T(\vartheta)\,d\vartheta]+F_1\int\limits_0^{\tilde{\mathrm{Fo}}}\bar{q}(\vartheta)\,d\vartheta\}\,d\tilde{\mathrm{Fo}}=F_2\int\limits_0^{\mathrm{Fo}}T(\mathrm{Fo}-\tilde{\mathrm{Fo}})\,T(\tilde{\mathrm{Fo}})\,d\tilde{\mathrm{Fo}}, \tag{9}$$

where

$$\bar{q}(\mathrm{Fo})=q(\mathrm{Fo})\,r_0/\lambda^*;\quad \mathrm{Fo}=a^*\tau/r_0^2;\ a^*=\lambda^*/c^*;$$

$$F_1=\frac{\lambda^* r_0^{k-1}}{\tilde{\lambda}_0\beta m}\,[\varphi_1(r_0)\,\varphi_2(r_0)]^{\frac{1}{m-2}};\quad F_2=\frac{\tilde{c}_0}{c^* r_0^{k+1}\,\beta m}\,[\varphi_1(r_0)\,\varphi_2(r_0)]^{\frac{m-1}{m-2}}.$$

Here λ^* and c^* are certain values of $\lambda(r)$ and $c(r)$ from the ranges of their variation (in the case of constant thermophysical properties, we can set $\lambda^* = \lambda$ and $c^* = c$). The parameter β for the laws of change of λ and c in accordance with (3) is respectively equal to $1 - l$, $\alpha(1 - l)$, and l. The functions φ_1 and φ_2 are in turn equal (for the laws of change in λ and c being considered) to r_0^l, r_0^n; $(1+\alpha r_0)^l$, $(1+\alpha r_0)^n$; $\exp(-lr_0)$, $\exp(-nr_0)$. We should note that for a semiinfinite rod Eq. (9) is valid only when λ, $c \neq 0$ and λ, $c \neq \infty$ at $r = 0$. In the general case, for a semiinfinite rod with an exponential change in λ and c in the space variable the following formula is valid:

$$\bar{q}(\mathrm{Fo}) = \frac{1}{\Gamma(\nu)(2-l+n)^{\gamma}} \int_0^{\mathrm{Fo}} (\mathrm{Fo} - \tilde{\mathrm{Fo}})^{-\nu} T'(\tilde{\mathrm{Fo}})\, d\tilde{\mathrm{Fo}}, \qquad \gamma = 2\nu - 1; \quad 0 < \nu < 1; \tag{10}$$

$\Gamma(\nu)$ is a gamma function. When $\nu = 1/2$, Eq. (10) coincides with the formula for a semi-infinite body with constant thermophysical properties. The condition $0 < \nu < 1$ is connected with the requirements for finiteness of the thermal resistance and heat capacity of the section near r = 0 [2].

It follows from an examination of (9) that the type of geometry of the temperature field and the character of change in the thermophysical properties in the space variable for the laws (3) affect only the values of the parameters f_1, ν, F_1, and F_2. This allows us to realize a universal algorithm for computing heat fluxes from measurements of nonsteady temperatures of the surface of bodies (fragments of a structure) of different geometry with different laws of change in λ and c in the space variable.

However, it should be kept in mind that (9) is an integral equation of the first type (nonlinear), so that in the general case the algorithm for numerical realization is unstable with respect to small errors in T and approximation and rounding errors. On the other hand, it is known that the recalculation of boundary conditions "is stable for fairly general assumptions regarding the input function" [3]. This makes it possible to realize a stable algorithm for calculating q with (9) while satisfying certain requirements regarding the computing scheme — mainly, having the same types of approximating formulas for T and q at the first j computing steps. Nearly the same agreement can be ensured by calculating q from algorithms obtained from (6), (7) after changing over to the originals with small Fo (large s), these algorithms being applicable to temperature fields of different geometries. Calculations show that adequate agreement of q and T is ensured up to Fo* = 0.2-0.3. This is evident from the results of solution of the model problem shown in Fig. 2, where the initial data ("experimental" ΔT) was the results of solution of the direct problem of the heating of a plate 5 mm thick with η = 40 W/(m·K), a = 0.04 m^2/h.

NOTATION

τ, time; r, coordinate; T, temperature; q, heat flux; λ, thermal conductivity; c, volumetric specific heat; a, diffusivity.

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